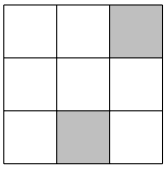
2701. Find the coordinates of the point at which the curve $xy^3 - 2x = 3$ is parallel to the y axis.

2702. State, with a reason, whether the following holds: "In a one-tailed test at the 5% level for the mean μ_0 of a normal distribution, the critical value c is such that, assuming H_0, $\mathbb{P}(\bar{X} - \mu_0 > c) = 0.05$."

2703. Prove that no cubic graph is convex everywhere.

2704. In the diagram, two squares of a three-by-three grid have been shaded:



Show that there are 24 different ways of shading two squares of a three-by-three grid such that the shaded squares do not share a border.

2705. Prove that, if parabolae $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ are tangent, then $\mathrm{f}(x) - \mathrm{g}(x)$ never changes sign.

2706. The fraction $\frac{84}{1763}$ can be expressed as the sum of the reciprocals of two consecutive odd numbers. Find them.

2707. The quartic approximation to $\cos\theta$, for small θ, is

$$\cos\theta \approx 1 - \tfrac{1}{2!}\theta^2 + \tfrac{1}{4!}\theta^4.$$

Find the percentage error at $\theta = \frac{\pi}{6}$ rad.

2708. You are given that

$$\log_8 2y = \log_2 x + \log_{16} x.$$

Find y in simplified terms of x.

2709. In this question, model the Earth as a fixed sphere of mass 5.97×10^{24} kg and radius 6370 km.

Newton's *law of universal gravity* states that the gravitational force between two masses m_1 and m_2 is given by

$$F = \frac{Gm_1m_2}{r^2},$$

where $G = 6.67 \times 10^{-11}$ $\text{Nkg}^{-2}\text{m}^2$ is a constant and r is the distance between the two masses.

(a) Show that, at the Earth's surface, this model predicts acceleration of 9.81 ms^{-2}, to 3sf.

(b) Find the predicted acceleration 10 km above the Earth's surface, to 3sf.

2710. Two samples of bivariate data $\{(x_i, y_i)\}$ each have strong +ve correlation, with coefficient $r \approx 0.8$. The combined sample, however, has coefficient $r \approx -0.5$. Explain, with reference to a sketched scatter diagram, how this is possible.

2711. Sketch $\sqrt{y} = x^2 - x$.

2712. A sector, with area, arc length, radius and angle denoted by the usual symbols, undergoes variation in time. Prove that

$$\frac{dA}{dt} = l\frac{dr}{dt} + \tfrac{1}{2}r^2\frac{d\theta}{dt}.$$

2713. A region, with area A, is enclosed by the curves $x = y^2$, $x = (12 - y)^2$ and the y axis.

(a) Sketch the region.

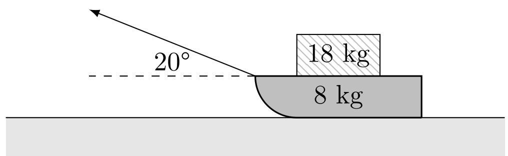
(b) Using integration, show that $A = 144$.

2714. Find the stationary point of $y(\ln x + 1) = e^x$.

2715. Solve $\left(\sqrt{x} + x\right)^4 + \left(\sqrt{x} - x\right)^4 = 4 + 12x^3$.

2716. The interior angles of a pentagon form an AP. Give, in radians, the set of possible values for the second smallest angle.

2717. A sledge of mass 8 kg, carrying a load of equipment of mass 18 kg, is being dragged into motion, with acceleration 0.5 ms^{-2}, by means of a rope angled at 20° above the horizontal. The ground is rough, as is the contact between sledge and load. The coefficient of friction between the sledge and the ground is 0.1. The coefficient of friction between the sledge and the load is large enough to avoid slippage. Other resistances may be neglected.



(a) Draw force diagrams for the load and for the combined sledge and load.

(b) Find the tension in the rope.

(c) Find the magnitude of the total contact force acting on the load.

2718. A hyperbola and a parabola have equations

$$x^2 - y^2 = 4,$$
$$4y = x^2.$$

Show that the curves are tangent.

2719. Use a substitution to show that

$$\int_0^4 \frac{1}{8 - \sqrt{x}} + 1\,dx = 16\ln\tfrac{4}{3}.$$

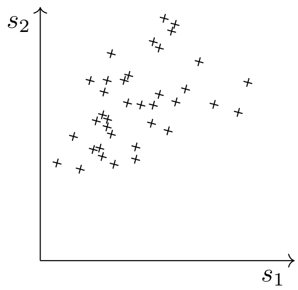
2720. A jar contains n red and $(100 - n)$ blue counters. Two counters are removed and observed. After the observation, they are returned to the jar. This is repeated many thousands of times. On average, 16 out of every 33 trials produces one of each colour. Find all possible values of n.

2721. The equation $x^2 - 3xy + y^2 = 1$ defines a hyperbola.

(a) Find $\dfrac{dy}{dx}$ in terms of x and y.
(b) Show that $3x + 7y = 24$ is normal to the curve.

2722. Find the acute angle between $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$.

2723. An exams faculty is analysing the scores s_1 and s_2 on two consecutive mathematics finals, among students at a university.



(a) Compare the performances on the two finals.
(b) Describe any correlation.
(c) The total mark is given by $s_1 + s_2$. Describe the correlation between the marks s_1 and s_2 among the ten highest-attaining students.
(d) Comment on your answer.

2724. Find and classify all stationary points of the curve

$$y = \frac{x+1}{x^2 + x + 1}.$$

2725. The quadratic functions f, g have discriminants which are zero and greater than zero respectively. They share no roots. For each of the following graphs, write down the number of roots and the number of vertical asymptotes:

(a) $y \cdot \mathrm{g}(x) = \mathrm{f}(x)$,
(b) $y \cdot \mathrm{f}(x) = \mathrm{g}(x)$,
(c) $y \cdot \mathrm{g}(x)^2 = \mathrm{f}(x)$.

2726. Find $\displaystyle\int \frac{2}{1 + \cos 2x}\, dx$.

2727. Prove that there are no constants A, B such that the following is an identity:

$$\frac{x^2 + x - 1}{x^4 + x^2} \equiv \frac{A}{x^2} + \frac{B}{x^2 + 1}.$$

2728. A *step function* $\mathrm{S}(x)$ is defined as

$$\mathrm{S}(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in [0, 1) \\ 0, & x \in [1, \infty). \end{cases}$$

Using $\bullet$ and $\circ$ to represent included or excluded points, sketch the following graphs:

(a) $y = \mathrm{S}(x)$,
(b) $y = x^2\,\mathrm{S}(x)$,
(c) $y = \mathrm{S}(2 - x)$,
(d) $y = \mathrm{S}(\mathrm{S}(x))$.

2729. If $\dfrac{d}{dx}(\sin x + \cos y) = 1$, show that $\dfrac{dy}{dx} = \dfrac{\cos x - 1}{\sin y}$.

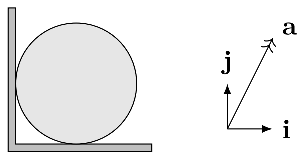
2730. A study of data has shown that the variables y^2/x and x are linearly related. When x is plotted on the usual horizontal axis and y^2/x on the vertical, the line has a negative gradient $-k$, for $k > 0$, and passes through $(2, 0)$.

(a) Show that $k(x - 1)^2 + y^2 = k$.
(b) Describe the locus of this relationship.

2731. Solve for n in $\displaystyle\sum_{k=1}^{n} (2k - 1) = 100$.

2732. The graph $y = x^4 - px^2$ has points of inflection at $x = \pm 1$. Determine the value of p.

2733. A tractor has picked up a hay-bale in its shovel. The tractor's shovel is modelled as consisting of rigid horizontal and vertical planes, and the bale as a cylinder of mass 200 kg.



The tractor lifts the bale in the direction shown, accelerating it at $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$ ms^{-2}. Relative to the shovel, the bale does not move.

(a) Explain, with reference to moments, why the frictional forces acting on the bale at the two points of contact must be equal in magnitude.
(b) Assume the shovel is smooth. Find the contact forces between the shovel and the bale.

2734. If $x = \sin^2 u$, show that $\dfrac{du}{dx} = \operatorname{cosec} 2u$.

2735. State, with a reason, whether the following holds: "In a binomial hypothesis test with $H_1 : p > 1/4$, if $\mathbb{P}(X \geq a) > 1/50$, then, at the 2% level, a lies in the acceptance region for the test."

2736. Prove that $\lim_{x\to 1} \dfrac{x^2-1}{x^3-1} = \frac{2}{3}$.

2737. True or false?

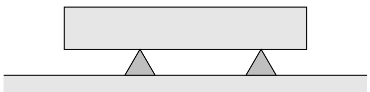
(a) There are 24 anagrams of ABCD.
(b) There are 12 anagrams of AABC.
(c) There are 8 anagrams of AABB.

2738. By separating the variables and integrating, find the general solution of the differential equation

$$\frac{dy}{dx} - 2y = 0.$$

2739. $\mathrm{F} : x \mapsto x^2 + 6x + 1$ and $\mathrm{G} : x \mapsto x^4 - px^2 - p$ have the same range over $\mathbb{R}$. Determine all possible values of the constant p.

2740. A uniform block of mass m kg rests on supports, as depicted. The supports divide the length of the block into sections in the ratio $p : q : r$.



Show that the supporting reaction forces differ by

$$R_1 - R_2 = \frac{p-r}{q} mg.$$

2741. A student claims: "If bivariate samples, each with strong negative correlation, are combined, then the combined sample necessarily has strong negative correlation." State, with a reason, whether this is true or not.

2742. Sketch the graph $y = -\cos(\arcsin x)$.

2743. Two numbers are said to be in the *golden ratio*, ϕ, if the ratio between them is the same as that of their sum to the larger of the two. Show that

$$\phi = \frac{\sqrt{5}-1}{2}.$$

2744. A curve C has equation $y = 2\left(1+x^2\right)^{-1} - 1$. Show that C has points of inflection at $x = \pm\frac{1}{\sqrt{3}}$.

2745. Let X represent the total number of successes in n independent trials of an experimental process, where each has a probability of success p. This information is summarised as $X \sim B(n, p)$.

(a) Explain, using a tree diagram or otherwise, why $\mathbb{P}(X = r) = {}^n\mathrm{C}_r p^r q^{n-r}$.
(b) Hence, prove that, for any $p \in [0, 1]$,

$$\sum_{i=0}^{i=n} \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i} = 1.$$

2746. Show that $\displaystyle\int_0^1 t(2\ln t + 1)\, dt = 0$.

2747. A chemical works produces gas in an industrial process. Rate of gas production, in cubic metres per second, is modelled, $t > 0$ seconds after the initiation of the process, with

$$\frac{dV}{dt} = (t^2 - t)e^{-t}.$$

(a) Show that, after initiation, the process begins absorbing gas.
(b) Find the time from which $\dfrac{dV}{dt} \geq 0$.
(c) Show that production peaks around $t = 2.6$.
(d) When production has become negligible, the process is terminated. Find the total volume of gas produced during the process.

2748. Show that $y = 2x - 5$ is an asymptote of

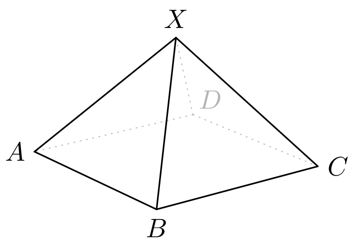
$$y = \frac{2x^2 - x - 7}{x+2}.$$

2749. A region is defined by those points simultaneously satisfying the inequalities

$$x^2 + (y+4)^2 \leq 4,$$
$$x^2 + (y+3)^2 \geq 1.$$

Find the area of the region.

2750. A square-based pyramid is shown below.



Each of the five faces of the pyramid is randomly and independently coloured either red, green or blue. Determine the probability that no two red faces share a border.

2751. In this question, do not use a calculator.

The graph $y = x^3 - 21x^2 + px - 315$, for some constant $p \in \mathbb{Z}$, crosses the x axis at three points which are in arithmetic progression.

(a) Explain, using symmetry, why the central root must be at a point of inflection.
(b) Hence, find p.
(c) Solve the equation.

2752. Explain why, if the two forces of a Newton III pair both act on the same rigid object, then those forces can have no effect on the behaviour of the object.

2753. Find the range over $\mathbb{R}$ of

(a) $\cos^2(ax+b)$,
(b) $a\cos^2(x+b)$,
(c) $a\cos^2 x + b$.

2754. Prove the change of base formula $\log_a b \equiv \dfrac{\log_c b}{\log_c a}$.

2755. A curve is defined by $y = \dfrac{x^2+x}{x-1}$.

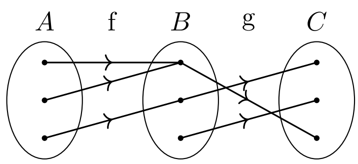
(a) Show that this curve has stationary points at $x = 1 \pm \sqrt{2}$.
(b) Classify these stationary points.

2756. A haulage truck is slowing down as it drives up a slope of 8° above the horizontal. Its acceleration is -0.2 ms^{-2}. Inside the truck, and not moving relative to it, is a pallet of mass 300 kg.

(a) Find the magnitude of the overall contact force exerted by the truck on the pallet.
(b) Find the acute angle above the horizontal at which this force acts.

2757. A polynomial function h has the property that, for all $a, b \in \mathbb{R}$ with $a > b$, $\mathrm{h}'(a) > \mathrm{h}'(b)$. State, with a reason, whether this implies that h is convex.

2758. Mappings f and g map between three sets A, B, C, as depicted below.



State whether each of the following mappings is, with the given domain and codomain, well-defined and/or invertible.

(a) $\mathrm{f} : A \mapsto B$,
(b) $\mathrm{g} : B \mapsto C$,
(c) $\mathrm{g} : A \mapsto C$,
(d) $\mathrm{gf} : A \mapsto C$,
(e) $\mathrm{g}^{-1} : C \mapsto B$,
(f) $\mathrm{f}^{-1}\mathrm{g}^{-1} : C \mapsto A$.

2759. The bell curve of the normal distribution satisfies the differential equation

$$\mathrm{f}'(x) = -\frac{x-\mu}{\sigma^2}\,\mathrm{f}(x).$$

Prove that $\mathrm{f}''(x) = 0$ at $x = \mu \pm \sigma$.

2760. Prove that there is no non-constant polynomial function f for which $\mathrm{f}(x) = \mathrm{f}(x+1)$ for all $x \in \mathbb{R}$.

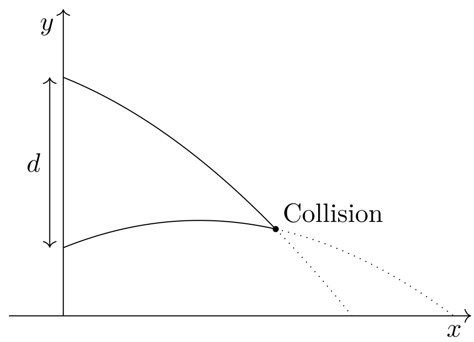
2761. A straight line segment is given parametrically, for $t \in [0, 2]$, by $x = 2t+1$, $y = t+4$.

(a) Sketch this line segment.
(b) Determine the area between this line segment and the x axis, using the formula

$$A = \int_{t_1}^{t_2} y \frac{dx}{dt}\, dt.$$

(c) Verify your answer using the formula for the area of a trapezium.

2762. Two particles are projected simultaneously, at the same speed u, from two points d metres vertically above one another. The particles collide in the subsequent motion.



(a) Show that the angles of projection must be θ above and θ below the horizontal.
(b) Show that collision takes place at a horizontal distance of $\frac{1}{2}d\cot\theta$ from projection.

2763. A graph has implicit equation $y - \dfrac{1}{x+y} = 0$.

(a) Find explicit equations with
  i. x as the subject,
  ii. y as the subject.
(b) Hence, show that the curve has asymptotes at
  i. $y = 0$,
  ii. $y = -x$.
(c) Show that no tangent to the curve is parallel to the y axis.
(d) Sketch the curve.

2764. Two dice are rolled. Given that the sum of the two scores is at least eight, find the probability that at least one score is a six.

2765. Show that $\mathrm{f}(x) = \ln\left(1+e^x\right)$ is convex everywhere.

2766. A rigid object has the following forces applied to it in a horizontal (x, y) plane:

$$\begin{aligned} &3\mathbf{i} \text{ N at } (0,1),\\ &2\mathbf{j} \text{ N at } (2,1),\\ &a\mathbf{i}+b\mathbf{j} \text{ N at } (1,2). \end{aligned}$$

Show that there are no values of a and b for which equilibrium will be maintained.

2767. Sketch the graph $y = \dfrac{|x|\,(4+x)}{x}$.

2768. Write the following in simplified terms of $\ln x$:

(a) $\ln(x^2)$,
(b) $\ln(e^2 \cdot x)$,
(c) $\ln(e^2 \div x)$.

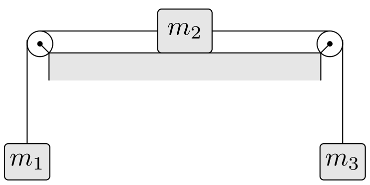
2769. The equation $x^4 - 4x^3 - 20x^2 + k = 0$ has no real roots. By considering stationary points, determine the possible values of k, answering in set notation.

2770. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a polynomial function f:

① division of $\mathrm{f}(x)$ by $(x - \alpha)$ leaves a non-zero remainder,
② $\mathrm{f}(\alpha) \neq 0$.

2771. Find $\displaystyle\int 2x\sqrt{x^2+4}\,dx$.

2772. A smooth pulley system of three masses is set up on a table as depicted below, where $m_3 > m_1$.



(a) Explain the assumptions necessary to model
i. the accelerations of the masses as equal,
ii. the tension as taking only two values.
(b) Making all necessary assumptions, prove that the acceleration a ms^{-2} is given by

$$a = \frac{(m_3 - m_1)g}{m_1 + m_2 + m_3}.$$

2773. Sketch $x^3 y = 1$.

2774. Consider $y = \log_k x$, for constant $k > 0$.

(a) Write the above in the form $x = e^{\mathrm{f}(y)}$.
(b) Hence, prove that $\dfrac{d}{dx}(\log_k x) = \dfrac{1}{x \ln k}$.

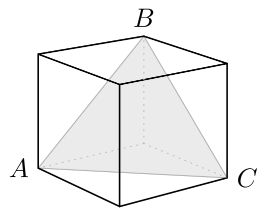
2775. Four values x_1, x_2, x_3, x_4 are chosen at random and independently from the interval $[0, 1]$. Write down

$$\mathbb{P}(x_1 < x_2 < x_3 < x_4 \mid \textstyle\sum x_i = 0.5).$$

2776. For each of the following, give the equation of the reflection of the graph in the line $y = x + k$:

(a) $y = x + k$,
(b) $y = x^2 + k$,
(c) $y = x^3 + k$.

2777. The diagram shows a cube, and a triangle ABC formed of three vertices:



Determine, in the exact form $\theta = \arctan k$, the acute angle θ between $\triangle ABC$ and the base.

(The angle between planes is defined as the angle between their normals.)

2778. Answer the mathematical joke: “Two cats jump onto a sloped roof. Which one slides off first?”

2779. Determine the area of the region of the (x, y) plane whose points simultaneously satisfy the following inequalities:

$$\begin{aligned} (x+y+1)^2 &< 1,\\ (x-y-1)^2 &< 1. \end{aligned}$$

2780. Prove that the quotient of two monic quadratic functions f and g may be expressed, for constants a, b, c, d, as

$$x \longmapsto 1 + \frac{ax+b}{x^2+cx+d}.$$

2781. The first, third and fifth terms of a GP are given as $2d - 3$, d, $2d + 3$. Find all possible values of d.

2782. A particle has position $\mathbf{r} = \begin{pmatrix} 1+t^3 \\ 1-2t^3 \\ 4+t^3 \end{pmatrix}$ m.

(a) Show that the magnitude of the acceleration increases linearly with time.
(b) Show that the particle accelerates without changing its direction of motion.

2783. Prove that the sum of three consecutive cubes greater than eight has at least five factors.

2784. In a party game, six people sitting around a round table simultaneously give either a thumbs up or a thumbs down. The choices are assumed to be random and independent.

(a) Find the probability that there are as many thumbs up as thumbs down.

(b) Given that there are as many thumbs up as thumbs down, determine the probability that everyone is sitting next to someone who has chosen the same result as them.

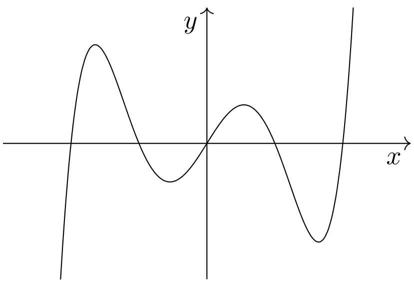
2785. For $\text{f} : x \mapsto \dfrac{\sin x + 1}{\sin x - 1}$, find f^{-1}.

2786. The major functional symmetries are:

① odd: rotation about O,

② even: reflection in $x = 0$.

The diagram shows a curve with odd symmetry.



Prove that, if $y = \text{f}(x)$ has odd symmetry, then $y = \text{f}'(x)$ has even symmetry.

2787. By first solving a boundary equation, determine $\mathbb{P}(X^2 + X > 10)$, where $X \sim N(5, 3^2)$.

2788. In graphics, a *pixel* is the basic building block of a visual display. Usually, and in this question, it is a small square.

A rectangular graphics element is enlarging in the x and y directions separately. Initially, it measures 240 by 360 pixels, and those lengths are increasing by 20 and 50 pixels per second respectively.

(a) Find the initial rate of change in area, in units of pixels per second.

(b) Explain, with reference to the units in part (a) and in the text above it, how usage of the unit "pixel" differs mathematically from usage of the unit "centimetre".

2789. Show that the following equation has exactly one real root:

$$x(3x + 2) = \frac{7}{x} - 20.$$

2790. Either prove or disprove the following statement: If four vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ sum to zero, then they must all lie in the same plane.

2791. Consider the graph $y = (2x - 1)^3 - 13(2x - 1)^2$.

(a) Explain how you know, without calculus, that the curve has a stationary point on the x axis.

(b) Write down the x coordinate of this point.

(c) Find the other x axis intercept.

(d) Sketch the curve. You don't need to find the other stationary point.

2792. State, with a reason, whether the curve $y = |x|$ intersects the following curves:

(a) $x = |y|$,

(b) $x = |y + 1|$,

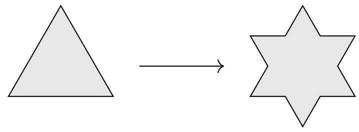
(c) $x = |y| + 1$.

2793. A *random walk* in 1D involves a sequence of steps along a number line, starting at zero, each of which, with equal probability, is a move of $+1$ or -1. In this question, five steps are made.

(a) Show that the probability of ending up at an even number is zero.

(b) Find the probability of ending up at -3.

2794. The *Koch snowflake* is a fractal shape produced by the iteration depicted below. At each stage, every line segment has an equilateral triangle added onto the middle third of its length. Shown are K_1 and K_2, the first and second versions of the snowflake. The area of K_1 is 1.



(a) Show that the boundary of K_n is made up of $3 \times 4^{n-1}$ line segments.

(b) Find, in terms of n, the area added when K_n becomes K_{n+1}.

(c) Hence, show that the area of K_n tends to 160% of the area of the original triangle.

2795. By the side of a road, traffic cones have been placed 5 metres apart. At time $t = 0$, an approaching car begins decelerating constantly, and passes the first three cones at $t = 1.410, 1.590, 1.775$ s. Determine the number of cones the car passes before coming to rest.

2796. A term u_k of the sequence $u_n = n^3 - n^2$ has value 63840000. Using a numerical method (but not a polynomial solver), find k.

2797. A curve has gradient

$$\frac{dy}{dx} = \frac{\ln a^x + \ln a}{\ln a^x - \ln a},$$

for some constant $a > 1$. Find y in terms of x, verifying that your equation is independent of a.

2798. A function h has domain $\mathbb{R}$ and range $[0, 2]$. State, with a reason, whether the following hold:

(a) $\big(\text{h}(x)\big)^2 \in [0, 4]$ for all $x \in \mathbb{R}$,

(b) $x \mapsto \big(\text{h}(x)\big)^2$ has range $[0, 4]$ over $\mathbb{R}$.

2799. A particle is projected from the point $(0, 3)$ with horizontal velocity 0.7 ms^{-1} and vertical velocity 1.4 ms^{-1}. Determine, in fully simplified form, the equation of the trajectory.

2800. By solving an equation and sketching graphs, solve the inequality $\sqrt{6 - x^2} > x^2$.

——— End of 28th Hundred ———